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Quantum Fluctuation in Thermal Vacuum State for a Mesoscopic RLC Circuit

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By means of the thermal field dynamics theory invented by Takahashi and Umezawa, we study the quantum effects of a nondissipative mesoscopic RLC circuit at a finite temperature.

KEY WORDS: quantum fluctuation; thermal vacuum state; mesoscopic; RLC circuit.

1. INTRODUCTION

Owing to the rapid development of nanometer techniques and microelectronics, the miniaturization of integrated circuits and components is going toward the atomic scale. Clearly, when the charge-carrier inelastic coherence length and the charge-carrier confinement dimension approach the Fermi wavelength, the classical physics is expected to be invalid and quantum effects must be taken into account. The quantum effects for a single LC lossless circuit was first discussed by Louisell (1973). Following a similar line of thougth, many authors have discussed the quantum effects of mesoscopic circuits (Chen *et al.*, 1995; Fan *et al.*, 1998; Li *et al.*, 1996; Wang *et al.*, 2000a,b; Zhang *et al.*, 1998). However, Louisell's result is obtained at $T = 0$, and the Joule heat effects have not been taken into account. It is well known that electric currents in a circuit (not superconductors) naturally produce Joule heat, and practical electric circuits are always working at a finite temperature. There is no doubt that the study of quantum noise of mesoscopic circuits at a finite temperature is very important both theoretically and experimentally. Recently, Fan *et al.* have studied the quantum fluctuation of a mesoscopic LC electric circuit at a finite temperature (Fan *et al.*, 2000). In this paper, we shall study the time evolution of a nondissipative mesoscopic RLC circuit, and employ thermal field dynamics (TFD) theory (Takahashi

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and Umezawa, 1975) to study the quantum fluctuation of the system at a finite temperature.

1.1. The quantization of a mesoscopic RLC circuit

For a nondissipative RLC circuits, the classical equations of motion, as a consequence of Kirchhoff's law, read (Wang *et al.*, 2000a)

$$L = \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \varepsilon(t), \tag{1}$$

where $q(t)$ is the charge of the RLC circuit, R, L, and C stand for the resistance, inductance, and capacitance, respectively, and $\varepsilon(t)$ is the electromotive force. The classical Hamiltonian is given by

$$H = \frac{p^2}{2L} + \frac{R}{2L}(qp + pq) + \frac{q^2}{2C} - q\varepsilon(t), \tag{2}$$

where $p = L\frac{dq}{dt}$. It is easily seen that Eq. (2) is analogous to a harmonic oscillator. According to the standard quantization method, we quantize the system by identifying q and p as Hermite operators and imposing the commutation relation $[q, p] = i$. Therefore Eq. (2) represents a quantized harmonic oscillator.

In order to diagonalize H, we introduce the following unitary operator U:

$$U = \exp\left(\frac{iR}{2}q^2\right). \tag{3}$$

By means of the operator formula

$$e^{\lambda X}Ye^{-\lambda X} = Y + \lambda[X, Y] + \frac{\lambda^2}{2!}[X, [X, Y]] + \dots, \tag{4}$$

we have

$$H' = U^{-1}HU = \frac{p^2}{2L} + \frac{1}{2}L\omega^2q^2 - q\varepsilon(t), \tag{5}$$

where $\omega = \omega_0\sqrt{1 - R^2C/L}$, $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency of an LC circuit without the resistance. Letting

$$a = \sqrt{\frac{L\omega}{2}}\left(q + \frac{i}{L\omega}p\right) \tag{6}$$

and

$$a^+ = \sqrt{\frac{L\omega}{2}}\left(q - \frac{i}{L\omega}p\right), \tag{7}$$

Eq. (5) can be rewritten as

$$H' = \omega\left(a^+a + \frac{1}{2}\right) + V(t)(a + a^+), \tag{8}$$

where

$$V(t) = -\frac{\varepsilon(t)}{\sqrt{2L\omega}}. \tag{9}$$

It can be proved that the time evolution operator $U_s(t, 0)$ corresponding to H' is given by (Fan, 1997)

$$U_s(t, 0) = \exp(-i\omega a^+ at)\exp\lfloor -i(\eta^* a^+ + \eta a)\rfloor. \tag{10}$$

Here we have omitted a phase factor,

$$\eta(t) = \int_0^t V(\tau)\exp(-i\omega\tau)\,d\tau \tag{11}$$

Therefore, the wave function of the system at time t is given by

$$|\psi(t)\rangle = UU_s(t, 0)|\psi(0)\rangle. \tag{12}$$

2. QUANTUM FLUCTUATION OF THE SYSTEM

We now study the quantum fluctuation of the system at finite temperature. The effect of temperature can be introduced in terms of the TFD theory, which was invented by Takahashi and Umezawa (1975). In TFD one associates (a, a^+), acting on a Hilbert space, with thermal freedom operators $(\tilde{a}, \tilde{a}^+)$ in the extended Hilbert space (a fictitious space). The operator $\tilde{a}$ and $\tilde{a}^+$ obey the following commutation relation:

$$[\tilde{a}, \tilde{a}^+] = 1, \qquad [\tilde{a}, a] = [\tilde{a}, a^+] = 0 \tag{13}$$

The ensemble average of an operator A defined in the original Hilbert space

$$\langle A\rangle = z^{-1}(\beta)\,\mathrm{tr}(A\,e^{-\beta H}), \quad z(\beta) = \mathrm{tr}(e^{-\beta H}), \tag{14}$$

can be calculated as a pure state (so-called thermal vacuum state) average, namely

$$\langle A\rangle = \langle 0(\beta)|A|0(\beta)\rangle, \tag{15}$$

where $\beta = (kT)^{-1}$ and k is the Boltzmann constant. Takahashi and Umezawa have proved that at finite temperature T the thermal vacuum state $|0(\beta)\rangle$ for a harmonic oscillator is given by (Takahashi and Umezawa, 1975)

$$|0(\beta)\rangle = S(\theta)|0\tilde{0}\rangle = \sec h\theta\,\exp(a^+\tilde{a}^+\tanh\theta)|0\tilde{0}\rangle, \tag{16}$$

where the thermal operator $S(\theta)$ is

$$S(\theta) = \exp\lfloor -\theta(a\tilde{a} - a^+\tilde{a}^+)\rfloor, \tag{17}$$

$$\tanh\theta = \exp\left(-\frac{\omega}{2kT}\right). \tag{18}$$

We now suppose the initial state of the system is in two-mode thermal vacuum state

$$|\psi(0)\rangle = |0(\beta)\rangle = S(\theta)|0\tilde{0}\rangle. \tag{19}$$

Then the wave function of the system at time t is given by

$$|\psi(t)\rangle = UU_s(t,0)|0(\beta)\rangle. \tag{20}$$

With the help of Eq. (4), we have

$$S^{-1}(\theta)aS(\theta) = a\cosh\theta + \tilde{a}^{+}\sinh\theta, \tag{21}$$

$$U_s^{-1}aU_s = a\,e^{iax} + z, \quad z = -i\eta^{*}(t)\,e^{-iax}, \tag{22}$$

$$U^{-1}pU = p - Rq. \tag{23}$$

From Eqs. (21)–(23) we obtain the averages and the mean-square values of the charge q and its conjugate variable p

$$\langle q\rangle = \sqrt{\frac{2}{\omega L}}\,\mathrm{Re}(z), \tag{24}$$

$$\langle p\rangle = \sqrt{2\omega L}\,\mathrm{Im}(z) - R\sqrt{\frac{2}{\omega L}}\,\mathrm{Re}(z), \tag{25}$$

$$\langle q^2\rangle = \frac{1}{2\omega L}[4\,\mathrm{Re}^2(z) + \cosh(2\theta)], \tag{26}$$

$$\langle p^2\rangle = \frac{1}{2}\omega L[4\,\mathrm{Im}^2(z) + \cosh(2\theta)] + \frac{R^2}{2\omega L}[4\,\mathrm{Re}^2(z) + \cosh(2\theta)] - 4R\,\mathrm{Re}(z)\,\mathrm{Im}(z). \tag{27}$$

Therefore, the fluctuations of the charge q and its conjugate variable p are

$$\langle(\Delta q)^2\rangle = \langle q^2\rangle - \langle q\rangle^2 = \frac{1}{2\omega L}\cosh(2\theta) = \frac{1}{2\omega_0\sqrt{L^2 - R^2LC}}\coth(\omega/2kT), \tag{28}$$

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{1}{2}\omega L\cosh(2\theta) + \frac{R^2}{2\omega L}\cosh(2\theta) = \frac{L}{2\omega_0 C\sqrt{L^2 - R^2LC}}\coth(\omega/2kT), \tag{29}$$

where $\omega_0 = 1/\sqrt{LC}$.

From Eqs. (24)–(29), we can draw the following conclusion:

(1) The averages of the charge q and its conjugate variable p depend only on parameters z, and have nothing to do with temperature T. Since the parameters z is related to the source $\varepsilon(t)$ by Eqs. (9) and (22), we can obtain the conclusion that the averages of the charge q and its conjugate variable p depend on the concrete form of the source $\varepsilon(t)$.
(2) The mean-square values of the charge q and its conjugate variable p are dependent on both the parameters z and temperature T.
(3) The quantum fluctuations of the charge q and its conjugate variable p are independent on the parameters z, i.e., they have nothing to do with the concrete form of the source $\varepsilon(t)$.
(4) From the property of function $\coth(x)$ we can see that the quantum noise of a nondissipative mesoscopic RLC circuit increases with the rising temperature T and resistance R (for fixed L and C).

Letting $R = 0$, we obtain the quantum fluctuations of the charge q and its conjugate variable p for an LC circuit

$$\langle(\Delta q)^2\rangle = \frac{1}{2\omega_0 L}\coth(\omega_0/2kT), \tag{30}$$

$$\langle(\Delta p)^2\rangle = \frac{1}{2\omega_0 C}\coth(\omega_0/2kT). \tag{31}$$

It is easily seen that the quantum fluctuations of the charge q and its conjugate variable p for an LC circuit are smaller than that for an RLC circuit.

3. CONCLUSION AND DISCUSSION

In short, we have studied the quantization of a mesoscopic RLC circuit and discussed its time evolution. By means of the TFD theory we have studied the quantum fluctuation of the system at a finite temperature. The results show that the quantum noise of a nondissipative mesoscopic RLC circuit increases with the rising temperature T and resistance R (for fixed L and C). Although both the averages and the mean-square values of q and p are dependent on the concrete form of the source $\varepsilon(t)$, the quantum fluctuations of q and p have nothing to do with the concrete form of the source $\varepsilon(t)$.

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